Blowing up the Projective Plane

Group actions & Cox rings

Problem: Let G be a group acting on $R = k[x_1 \dots x_n]$. Is
$R^G = \{x \in R \mid gx = x \ \forall g \in G\}$ finitely generated?

Nagata 58': No in general. Let $S = k[x_1 \dots x_r, y_1, \dots y_r]$
& Consider the action of $\mathbb{G}_a^r$ (= The additive group $\mathbb{A}^r$)

$$\begin{cases} t_i \cdot x_i = x_i \\ t_i \cdot y_i = y_i + t_i x_i \end{cases}$$

Let $G \subset \mathbb{G}_a^r$ be a general linear subspace of codim ≥ 3
and $\dim G = g = 13$

$\boxed{\text{(Nagata): } S^G \text{ not finitely generated.}}$

Steinberg: $g = 6 \Rightarrow S^G$ not finitely generated. (we will see this)

Definition (We assume $\mathrm{Pic}(X)$ is torsion free)

Let X be projective and $D_1 \dots D_r$ a generating set for $\mathrm{Pic}(X)$.

The <u>Cox ring</u> of X is

$$\mathrm{Cox}(X) = \bigoplus_{(m_1 \dots m_r) \in \mathbb{Z}^r} H^0(X, m_1 D_1 + \dots + m_r D_r)$$

This is graded by $\deg S = D$ if $S \in H^0(X, D)$.

X is called a <u>Mori dream space</u> if $\mathrm{Cox}(X)$ is finitely generated.

Example $X = \mathbb{P}^2 = \mathrm{Proj}\, K[x_0, x_1, x_2]$

$\mathrm{Pic}(X) = \mathbb{Z}H$ for $H \subset \mathbb{P}^2$ a hyperplane.

$$\mathrm{Cox}(X) = \bigoplus_{n \geq 0} H^0(X, nH) \simeq K[x_0, x_1, x_2]$$

$\deg x_i = 1.$

Example $X = \mathrm{Bl}_1 \mathbb{P}^2$

$\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E$ where H is pullback of hyperplane and E is the exceptional.

$$\mathrm{Cox}(X) = \bigoplus_{m_1, m_2 \in \mathbb{Z}} H^0(X, m_1 H + m_1 E) = K[\underset{H}{x}, \underset{E}{y_1}, \overset{H-E}{z_1}, \overset{H-E}{z_2}]$$

Remark $\mathrm{Cox}(X)$ is the homogeneous coordinate ring of the toric variety X.

Fact! (Nagata) For G as above we have

$$S^G \simeq \mathrm{Cox}\left(\mathrm{Bl}_r \mathbb{P}^{r-g-1}\right)$$

In particular: $g=6$, $r=9$ gives $S^G \simeq \mathrm{Cox}(\mathrm{Bl}_9 \mathbb{P}^2)$

Definition

A (-1)-curve on a projective surface X is an irreducible curve $C \subset X$ such that $C^2 = -1$ & $C \simeq \mathbb{P}^1$.

Prop If X contains infinitely many (-1)-curves, then X is not a Mori dream space.

Proof: It suffices to see that the cone of effective divisors is not finitely generated.

$$\mathrm{Eff}(X) = \left\{ \sum a_i D_i \mid D_i \in \mathrm{Pic}(X) \text{ effective}, a_i \geq 0 \right\} / \equiv$$

Suppose $\mathrm{Eff}(X)$ is generated by $C_1 \dots C_r$. Then for a (-1)-curve C we have

$$C = \sum_{i=1}^r a_i C_i \qquad \text{for } a_i \geq 0.$$

Then $-1 = C^2 = \sum_{i=1}^r a_i (C \cdot C_i) \Rightarrow C \cdot C_i < 0$ for some i

$\Rightarrow C \subset C_i$ for some i.

Goal

- Compute (-1)-curves of $Bl_r \mathbb{P}^2$ & Connect to Weyl groups & dynkin diagrams.
- $Bl_9 \mathbb{P}^2$ is not a Mori dream space.
- What about $Bl_r \mathbb{P}^2$ $r \geq 10$?

Some Observations

Let $X = Bl_r \mathbb{P}^2$. H hyperplane class, E_i the exceptional divisor.

1) $Pic(X)$ is freely generated by $H, E_1, \dots, E_r$

2) The canonical divisor is $-K_X = 3H - E_1 - \dots - E_r$

3) Any curve class C not an exceptional divisor is of the form

$$C = dH - \sum a_i E_i \qquad a_i \geq 0.$$

$H \cdot E_i = 0 \qquad H^2 = 1 \qquad E_i^2 = -1 \qquad E_j \cdot E_i = 0 \quad i \neq j.$

A bound for d

- Suppose C is a (-1)-curve and not an exceptional divisor.

Write
$$C = dH - \sum a_i E_i$$

$\rightarrow C^2 = -1 \Rightarrow \quad d^2 - \sum a_i^2 = -1$

$\rightarrow$ (RR) $\; 2g - 2 = K_X \cdot C + C^2 \Rightarrow \quad \sum a_i = 3d - 1$

$\rightarrow \left(\sum_{i=1}^{r} a_i\right)^2 \leq r \sum_{i=1}^{r} a_i^2 \quad (r-9)d^2 + 6d + (r-1) \geq 0$

$$\Downarrow$$

For $r \leq 8$ we have only a finite number of (-1)-curves.

r	1	2	3	4	5	6	7	8
$d \leq$	0	1	1	1	2	2	3	7

The dynkin diagram of a blowup $\quad r \geq 3$

$X = Bl_r \mathbb{P}^2 \qquad H$ hyperplane $\qquad E_1, \dots, E_r$ exceptional divisors.

$K_X = 3H - \sum_{i=1}^{r} E_i \,. \qquad V = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \qquad (r+1)$-dim vector space.

The intersection product is a quadratic form on V.

⇝ Let $E_N = (K_X)^{\perp}$ Orthogonal complement w.r.t intersection product.

⇝ Basis for E_N: $\quad \alpha_1 = E_1 - E_2 \; \dots \; \alpha_{r-1} = E_{r-1} - E_r$

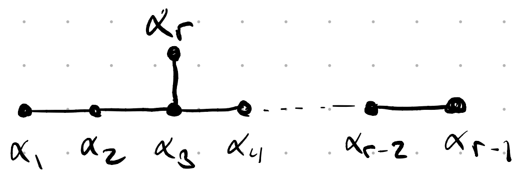
$$\alpha_r = H - \sum_{i=1}^{\cancel{r}\,3} E_i = H - E_1 - E_2 - E_3$$

⇝ Let $\alpha_i^{\vee}$ denote the dual basis and define

$$T_i(x) = x + \alpha_i^{\vee}(x) \cdot \alpha_i \qquad \text{for} \qquad x \in \mathrm{Pic}(X) \otimes \mathbb{R}$$

Def The group group generated by the T_i is denoted $W(E_n)$ and called the Weyl Group.

The α_i gives a dynkin diagram and $W(E_r)$ is the reflection group.



(Type E_r)

Fact (Nagata)

There is a 1:1 correspondance between (-1)-curves on X and the orbit of an exceptional divisor under the action of the weyl group.

$Bl_1 \mathbb{P}^2$

One (-1)-curve: E itself. $d \leq 0$

$Bl_2 \mathbb{P}^2$

$d \leq 1$

$\operatorname{Pic}(X, E_1, E_2) \rightsquigarrow$ E_1 & E_2 are (-1)-curves.

$(H-E_1-E_2)^2 = -1$ is also one. No more.

$$\{ E_1, E_2, H-E_1-E_2 \}$$

$Bl_3 \mathbb{P}^2$ $\quad d \leq 1$

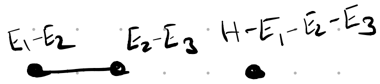
$\mathrm{Pic}(X) = \langle H, E_1, E_2, E_3 \rangle$

The (-1)-curves are:

$$E_1 \quad E_2 \quad E_3$$

$$H - E_i - E_j \qquad i, j \in \{1, 2, 3\} \text{ distinct.}$$

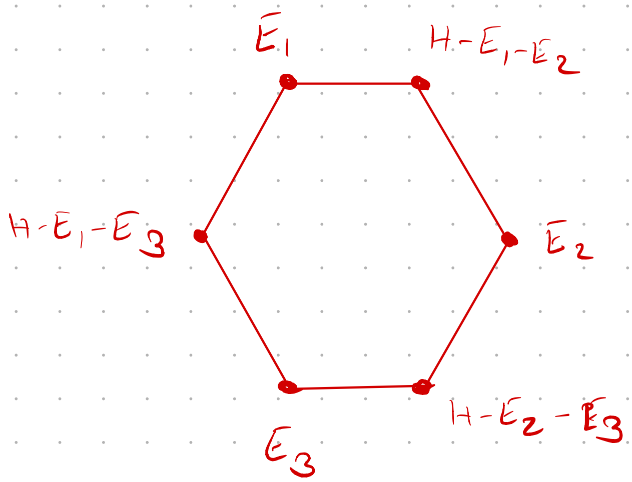
Total: $3 + \binom{3}{2} = 6.$

Dynkin diagram 

(Classically called $A_2 \times A_1$)

$|W(E_3)| = 12$

Intersection graph



Automorphism group: D_6 (order 12)

$Bl_4 \mathbb{P}^2$ $d \leq 1$

$\mathrm{Pic}(X) = \langle H, E_1, \dots, E_4 \rangle$

(-1)-curves $E_1 \dots E_4$

$H - E_i - E_j \qquad i \neq j \in \{1 \dots 4\}$

Total: $4 + \binom{4}{2} = 10$

Dynkin diagram

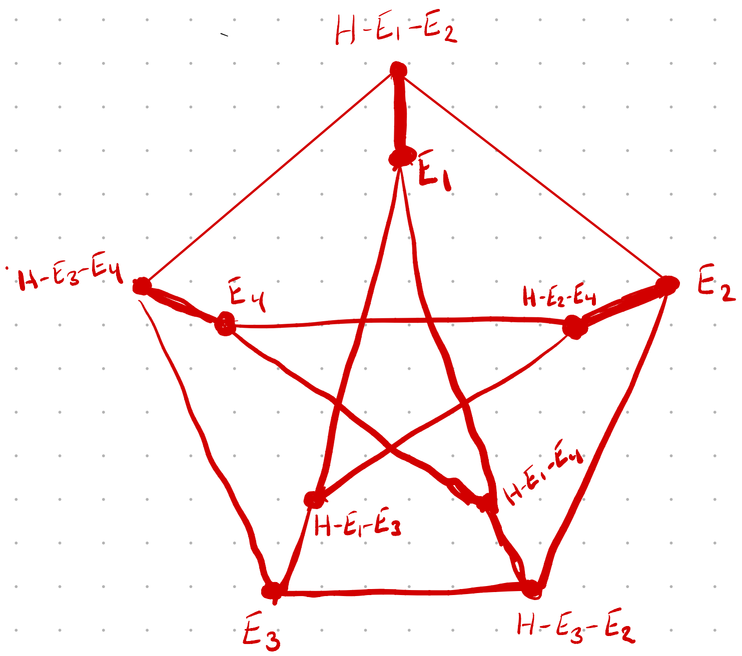
$E_1 - E_2$ — $E_2 - E_3$ — $E_3 - E_4$ — $H - E_1 - E_2 - E_3$

(Classically A_4)

$|W(E_4)| = 120$

Intersection Graph

Auto group S_5. Order 120.



$Bl_5\,\mathbb{P}^2$ $\qquad d \leq 2$

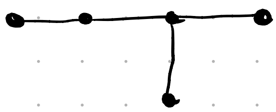
$\mathrm{Pic}(X) = \langle H, E_1, \dots, E_5 \rangle$

(-1)-curves $\qquad E_1 \dots E_5$

$\qquad H - E_i - E_j \qquad i \neq j \in \{1, \dots, 5\}$

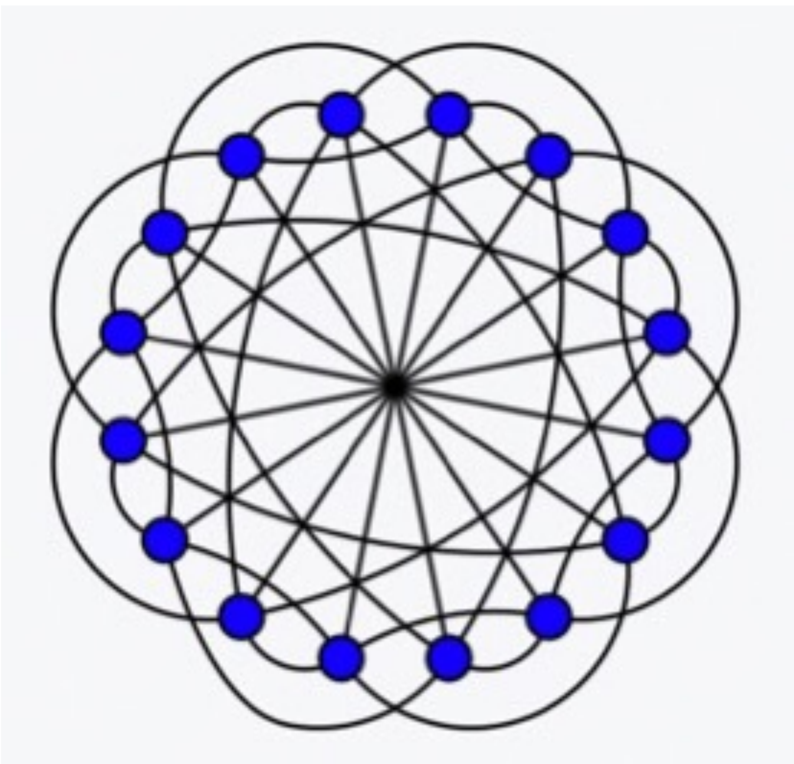
$\qquad 2H - E_1 - \dots - E_5$

Total: $5 + \binom{5}{2} + 1 = 16$

Dynkin diagram  (Also called D_5)

$|W(E_5)| = 1920$

Intersection graph



Automorphism group order 1920.

$Bl_6\, \mathbb{P}^2$

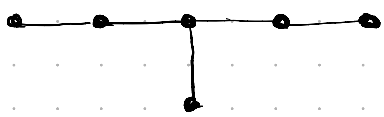
$\mathrm{Pic}(X) = \langle H, E_1, \dots, E_6 \rangle.$

(-1)-curves

	$E_1 \dots E_6$	6
	$H - E_i - E_j$	$\binom{6}{2}$
	$2H - E_{i_1} - \dots - E_{i_5}$	$\binom{6}{5}$

Total: $6 + \binom{6}{2} + \binom{6}{5} = 27 \longleftarrow$ Cubic

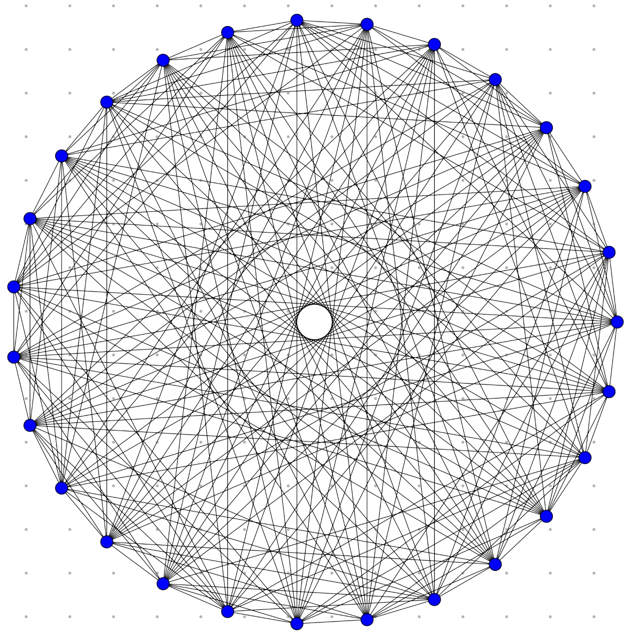
Dynkin diagram



E_6

$|W(E_6)| = 51840$

Intersection graph



Auto grp order 51840.

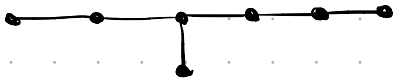
$Bl_7 \mathbb{P}^2$ $\quad d \leq 3$

$\mathrm{Pic}(X) = \langle H, E_1, \dots, E_7 \rangle$

(-1)-curves

	$E_1 \dots E_7$	7
	$H - E_i - E_j$	$\binom{7}{2}$
	$2H - E_{i_1} - \dots - E_{i_5}$	$\binom{7}{5}$
	$3H - 2E_{i_1} - E_{i_2} - \dots - E_{i_7}$	$\binom{7}{6}$

Total = 56

Dynkin diagram  E_7

$|W(E_7)| = 2903040$

$Bl_8\, \mathbb{P}^2$ $\quad d \leq 7$

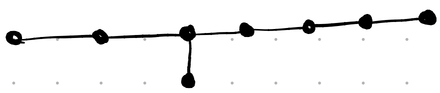
$\mathrm{Pic}(X) = \langle H, E_1, \dots, E_8 \rangle$

(-1)-curves

$E_1 \dots E_8$	8
$H - E_i - E_j$	$\binom{8}{2}$
$2H - E_{i_1} - \dots - E_{i_5}$	$\binom{8}{5}$
$3H - 2E_{i_1} - E_{i_2} - \dots - E_{i_6}$	$2 \cdot \binom{8}{6}$
$4H - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - E_{i_4} - \dots - E_{i_8}$	$\binom{8}{3}$
$5H - 2E_{i_1} - \dots - 2E_{i_6} - E_{i_7} - E_{i_8}$	$\binom{8}{2}$
$6H - 3E_{i_1} - 2E_{i_2} - \dots - 2E_{i_8}$	$\binom{8}{7}$

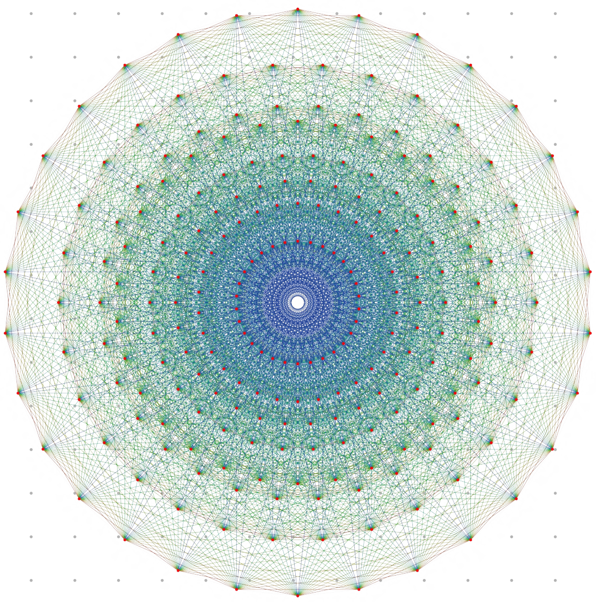
Total: 240

Dynkin diagrams



E_8

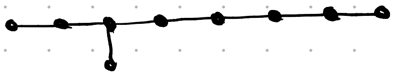
$|W(E_8)| = 696729600$



$Bl_9 \mathbb{P}^2$

$Pic(X) = \langle H, E_1, \dots, E_9 \rangle$

⚠ No longer a bound on d

⚠ The dynkin diagram E_9 

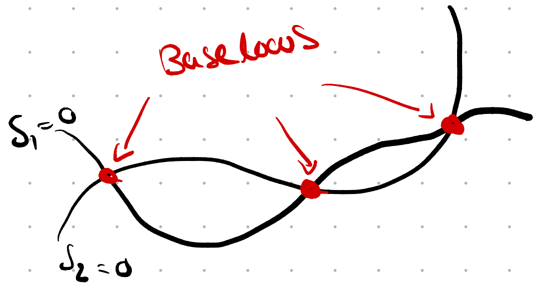
The weyl group $W(E_9)$ is infinite!

X has infinitely many (-1)-curves.

Consider very general sections $s_1, s_2 \in H^0(\mathbb{P}^2, \mathcal{O}(3))$

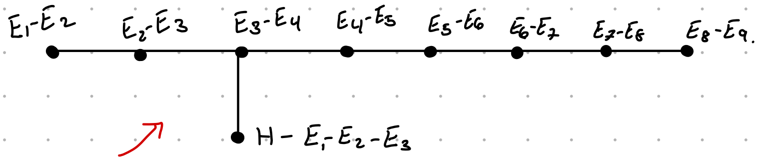
$\rightsquigarrow$ Pencil $as_1 + bs_2$ of cubics in $\mathcal{O}(3)$ with base locus consisting of 9 pts.

$X =$ Blowup of $\mathbb{P}^2$ in these 9 pts.



X contains infinitely many (-1)-curves.

Proof (Mukai) $\quad n=2 \quad r=9$



E_9

<u>Weyl group $W(E_9)$ generated by:</u>

For $i \in \{1 \dots 8\}$

$$T_i(E_i) = E_{i+1} \qquad T_i(E_{i+1}) = E_i$$

&

$$T_9(H) = 2H - E_1 - E_2 - E_3$$

$$T_9(E_j) = H - \sum_{\substack{i=1 \\ i \neq j}}^{3} E_i \qquad j = 1,2,3$$

$$T_9(E_i) = E_i \qquad i \geq 4.$$

<u>Recall:</u>

$$\{(-1)\text{-curves}\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{orbit of an } E_i \\ \text{under weyl grp}\end{array}\right\}$$

- For $\alpha = dH + a_1E_1 + \dots + a_9E_9$ write $\deg \alpha = d$.
- For $w \in W$ $\quad w(K_X) = K_X$. $\quad -K_X = 3H - E_1 - \dots - E_9$. In particular:

$$\deg w(-K_X) = 3 \deg w(H) - \sum_{i=1}^{9} \deg w(E_i) = 3 = \deg(-K_X)$$

<u>Claim</u>: Given $w \in W$ There is a subset $I \subset \{1, \dots, 9\}$ with $|I| = 3$ &

$$\sum_{i \in I} \deg w(E_i) \leq \frac{1}{3} \sum_{i=1}^{n} \deg w(E_i)$$

Take I to be the indices where $\deg w(E_i)$ is minimal.

Thm

$$\sum_{i \in I} \deg w(E_i) = \frac{1}{3} \sum_{i \in I} 3 \deg w(E_i) \leq \frac{1}{3} \sum_{i=1}^{9} \deg w(E_i)$$

Replace some of the $\deg w(e_i)$ with a larger number.

Let $\alpha_I = H - \sum_{i \in I} E_i$ then:

$$\deg(w(\alpha_I)) = \deg w(H) - \sum_{i \in I} \deg w(E_i)$$

$$\geq \deg w(H) - \frac{1}{3} \sum_{i=1}^{9} \deg w(E_i)$$

$$= \deg w(H) - \frac{1}{3}\left(3 \deg w(H) - 3\right)$$

$$= \deg w(H) - \deg w(H) + 1$$

$$= 1$$

Note: There is a reflection $R_I \in W(E_9)$ s.t

$$R_I(H) = 2H - \sum_{i \in I} E_i \quad (\text{Compose } T_9 \text{ \& suitable } T_i)$$

$\Rightarrow \; R_I(H) - H = \alpha_I$

$\Rightarrow \; \deg w(R_I(H)) - \deg w(H) = \deg w(\alpha_I)$

$\Rightarrow \; \deg w(R_I(H)) = \deg w(H) + 1$

This means: Given any $w(H)$ we can construct a new element $w \circ R_I$ in the weyl group such that $(w \circ R_I)(H)$ increases in degree

$\Rightarrow$ Orbit of H infinite

$\Rightarrow$ Orbit of E_i infinite!

$\Rightarrow$ Infinite nr of (-1) curves.

This also shows that $W(E_9)$ is infinite.

Remark

1: This shows that $W(E_9)$ is infinite!

2: Much more general statements are valid.

Alternative (Nagata)

Step 1: $-K_X$ defines a map $X \xrightarrow{f} \mathbb{P}^1$ Elliptic fibration.

Step 2: The exceptional divisors are sections of f.

Step 3: Using group structure of the fibers induce (in general) an infinite family of (-1)-curves.

$Bl_{10}\,\mathbb{P}^2$

$\mathrm{Pic}(X) = \langle H, E_1, \dots, E_{10} \rangle$

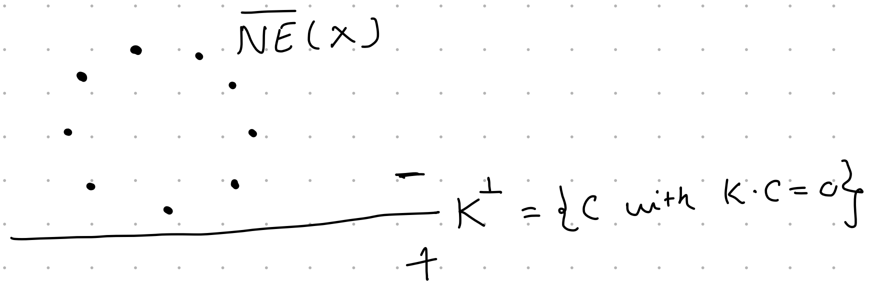
$-K_X = 3H - E_1 - \dots - E_{10}$

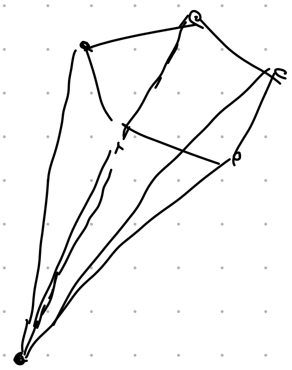
Note! $(-K_X)^2 = 9 - 10 = -1$

Some final observations

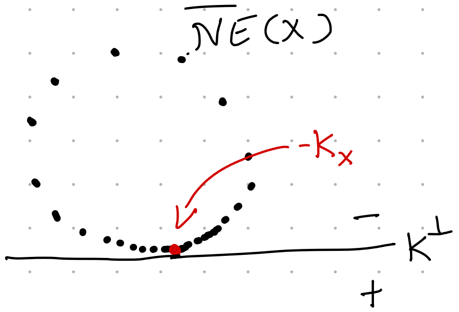
$\overline{NE}(X) = \overline{\{\text{Cone of curve classes } C \subset X\}} / \text{numerical equiv}$

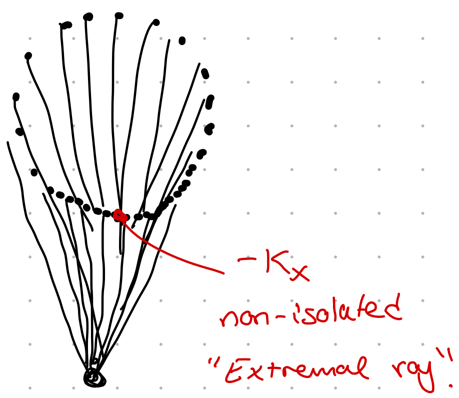
$r < 9$ $\quad -K_X$ ample and $\overline{NE}(X)$ is a rational polyhedral cone generated by rays of (-1) curves.



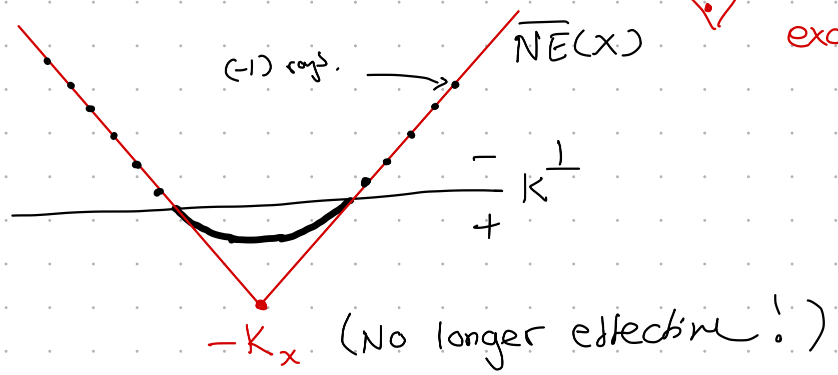


$r = 9$ $\quad$ Infinite number of (-1) curves. (Still isolated but accumulates)





$r = 10$



⚠ Conjectured that this is exactly $\overline{NE}(X)$.

The big difference is that the cone $\overline{NE}(X)$ becomes infinitely generated when r is large. For $r \geq 10$ you need an uncountably large generating set.